2023-24 MATH2048: Honours Linear Algebra II

Homework 9 Answer

Due: 2023-11-27 (Monday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. Let V be a finite-dimensional inner product space, and let T be a linear operator on V. If T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

   Proof. Given that T is invertible, there exists a unique linear operator T^{-1} such that $TT^{-1} = T^{-1}T = I$, where I is the identity operator.

   We want to show that T^* is also invertible and that $(T^*)^{-1} = (T^{-1})^*$.

   First, note that under the adjoint operation, we have $(TT^{-1})^* = (T^{-1})^*T^* = I^* = I$ and $(T^{-1}T)^* = T^*(T^{-1})^* = I^* = I$. Hence, $T^*(T^{-1})^* = (T^{-1})^*T^* = I$.

   This shows that there exists a unique operator $(T^{-1})^*$ such that $T^*(T^{-1})^* = (T^{-1})^*T^* = I$, which proves that T^* is invertible and its inverse is $(T^{-1})^*$, i.e., $(T^*)^{-1} = (T^{-1})^*$.

   This completes the proof. □

2. Let V be an inner product space, and let T be a linear operator on V. Prove the following results.

   (a) $R(T^*)^\perp = N(T)$.

   (b) If V is finite-dimensional, then $R(T^*) = N(T)^\perp$

   Proof. (a) If $x \in R(T^*)^\perp$, then $\langle x, T^*(y)\rangle = 0$ for any $y \in V$. So $\langle T(x), y\rangle = \langle x, T^*(y)\rangle = 0$ for any $y \in V$ which implies $T(x) = 0$ i.e. $x \in N(T)$.

   If $x \in N(T)$, then $\langle x, T^*(y)\rangle = \langle T(x), y\rangle = \langle 0, y\rangle = 0$ for any $y \in V$. So $x \in R(T^*)$.

(b) If V is finite-dimensional, then $R(T^*)$ is finite-dimensional. Therefore $R(T^*) = (R(T^*)^\perp)^\perp = N(T)^\perp$

$\square$

3. Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V. If W is T-invariant, then W is also T^*-invariant.

Proof. We know that since T is a normal operator, it is diagonalizable. Hence, $T|_W$, the restriction of T on W, is normal too and thus diagonalizable as well.

Let $\{w_1, w_2, \ldots, w_n\}$ be a basis for W consisting of eigenvectors of T. Since T is normal, its eigenvectors are also eigenvectors for T^*. This means $\{w_1, w_2, \ldots, w_n\}$ is also a basis for W consisting of eigenvectors of T^*.

Let $w \in W$ be arbitrary. Then w can be written as a linear combination of the basis vectors, say $w = \sum_{i=1}^n a_i w_i$ for some scalars a_i. Then $T^*w = \sum_{i=1}^n a_i T^* w_i = \sum_{i=1}^n a_i \lambda_i w_i$ where λ_i are the eigenvalues corresponding to the eigenvectors w_i of T^*. Hence, $T^*w \in W$ for all $w \in W$, meaning W is T^*-invariant.

This completes the proof. $\square$

4. Let T be a normal operator on a finite-dimensional inner product space V. Then $N(T) = N(T^*)$ and $R(T) = R(T^*)$.

Proof. Recall that an operator T is normal if $TT^* = T^*T$. Also recall that $N(T)$ denotes the nullspace (or kernel) of T and $R(T)$ denotes the range (or image) of T.

(i) We'll show that $N(T) = N(T^*)$:

Since T is normal, one has $\|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle T^*T(x), x\rangle = \langle TT^*(x), x\rangle = \langle T^*(x), T^*(x)\rangle = \|T^*(x)\|^2$ for any $x \in V$. So $x \in N(T) \iff \|T(x)\| = 0 \iff \|T^*(x)\| = 0 \iff x \in N(T^*)$. Therefore, we have $N(T) = N(T^*)$.

(ii) We'll show that $R(T) = R(T^*)$:

- Claim 1: $N(TT^*) = N(T)$.

  If $TT^*(x) = 0$, then $0 = \langle TT^*(x), x\rangle = \langle T^*T(x), x\rangle = \langle T(x), T(x)\rangle$ which implies $T(x) = 0$. If $T(x) = 0$, then $TT^*(x) = T^*T(x) = T^*(0) = 0$.

- Claim 2: $R(TT^*) = R(T)$.

  First, $R(TT^*) \subset R(T)$ is obvious. Second, by claim 1 and the rank-nullity theorem, one has $\text{rank}(TT^*) = \text{rank}(T)$. Therefore $R(TT^*) = R(T)$..

Thus $R(T) = R(TT^*) = R(T^*T) = R(T^*T^{**}) = R(T^*)$

Or alternatively, using the result of §6.3 Q12, one has $R(T^*) = N(T)^\perp = N(T^*)^\perp = R(T^{**}) = R(T)$.

This completes the proof. $\square$

5. Let U be a unitary operator on an inner product space V, and let W be a finite-dimensional U-invariant subspace of V. Prove that

   (a) $U(W) = W$

   (b) $W^\perp$ is U-invariant.

Proof. (a) If $x \in N(U|_W)$, then $0 = \|U|_W(x)\| = \|U(x)\| = \|x\|$, which implies $x = 0$. Therefore, $U|_W$ is one-to-one. Since $U|_W : W \to W$ is defined on a finite-dimensional space W, one has $U|_W$ is onto. Thus $U(W) = U|_W(W) = W$.

(b) Let $x \in W^\perp$. For any $y \in W$, by (a), there exists $z \in W$ such that $U(z) = y$. Then $\langle U(x), y\rangle = \langle U(x), U(z)\rangle = \langle x, z\rangle = 0$. Therefore $U(x) \in W^\perp$.

This completes the proof. $\square$